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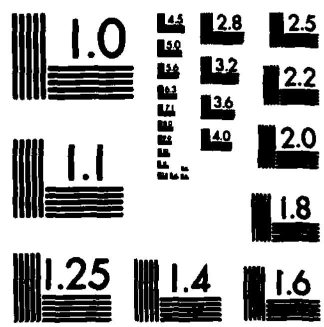
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Research Report CCS 486

ORDERED INCIDENCE GEOMETRY AND THE GEOMETRIC FOUNDATIONS OF CONVEXITY THEORY

by

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ORDERED INCIDENCE GEOMETRY AND THE

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Abstract

An Ordered Incidence Geometry, that is a geometry with certain axioms of incidence and order, is proposed as a minimal setting for the fundamental convexity theorems, such as the hyperplane separation theorem and the theorems of Radon and Helly. These theorems are usually stated, proved, understood and/or applied in the context of a linear vector space, but they require only incidence and order, (and for separation, completeness), and none of the linear structure of a vector space.

Key Words

Geometry: Axioms
Incidence and Order
Convexity and Inequalities
Convex Sets
Convex Analysis
Generalized Convexity
Separation Theorems
Radon and Helly Theorems

INTRODUCTION.

For motivation. consider the following quote:

> " The notion of separation has proved to be one of the most fertile notions in convexity theory and its applications. It is based on the fact that a hyperplane in R^n divides R^n evenly in two. in the sense that the complement of the hyperplane is the union of two disjoint open convex sets, the open half-spaces associated with the hyperplane.", ([19], p. 95).

Let us examine this statement by means of the following model of a plane geometry, the *Moulton Incidence Plane*, ([14], p. 58). *Points* are pairs (x,y) on the Cartesian plane, *lines* are either (i) vertical lines, or (ii) lines with non-positive slope, or (iii) *bent* lines of the form,

$$y = \begin{cases} mx + \beta \quad , & \text{if } x \leq 0 \\ \frac{m}{2}x + \beta \;, & \text{if } x > 0 \end{cases}$$

where $m > 0$. Through any two given points a, b in the plane, there passes a unique Moulton line $\overline{ab}$. Let the *interval* (a, b) between two points a, b be the set of those points in $\overline{ab}$ which lie between a and b (in the usual sense of *betweeness.*) A *convex set* can then be defined, as usual, as a set which with any two points a, b contains the whole interval (a, b). Does a Moulton line divide the plane into two convex sets?. The answer is "yes", its verification left to the reader. Does it follow then, as in the above quote, that two disjoint (Moulton) convex sets can be separated by a (Moulton) line?

Again the answer is "yes" (see the separation theorem (T7) below), but this cannot be deduced from classical convexity theory, which relies on the (linear) structure of the Euclidean vector space: The *Moulton plane* is not a model of a

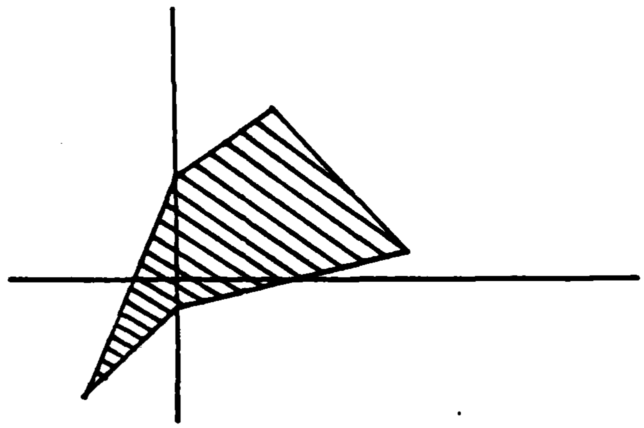

Fig 0.1 . A (Moulton) convex set

Euclidean geometry. Even if an ad-hoc proof can be found to establish a separation theorem for *Moulton convexity,* a fresh start would be needed in order to prove a similar result for "convex sets" in a plane geometry in which the (non-vertical) lines are given by

$$y = \cosh(\alpha + x) + \beta$$

The following issue thus arises: What are the *relevant* underlying assumptions (or axioms) for, say. a separation theorem? The structure of a metric vector space (or even a topological vector space) is not germane for separation, see e.g. [20] where the separation theorem is derived without topology. Here we go further, and discard with the *linear space structure* altogether. All our axioms are purely geometric. Indeed the ingredients of the standard separation theorem are the following notions: *Convex sets, hyperplanes* and *sides* of a hyperplane. These notions are purely geometrical, and can be fully described in terms of the following *primitives: Affine sets,* their *incidence* properties and *order* relations. The geometry introduced here is given exactly in terms of the above primitives. Its name, *Ordered Incidence Geometry,* is justified by the fact that for two or three dimensions, our axioms are similar to the first two groups of Hilbert's axioms for Euclidean geometry, namely the *incidence axioms* and the *order axioms,* [11]. We wish to single out one of these axioms, stating that:

"If a line, not passing through any of the vertices of a triangle, meets the triangle in one of its sides, then it must meet it in another side."

This postulate, introduced in 1882 by M. Pasch and known as the *Pasch Axiom*, was not included in Euclid's *Elements*. An interesting fact (well known to geometers) is that the Pasch axiom is *equivalent* to the following (separation) statement:

"A line [plane] divides the plane [space] into disjoint complementary convex sets."

Indeed, the Pasch axiom is the corner stone of the separation theorems in ordered incidence geometry.

For other generalizations of convexity theorems, under different axioms see e.g. the survey by Danzer, Grunbaum and Klee ([7]) and in particular, Ky Fan's generalization of the Krein-Milman Theorem ([9]), the Helly-type Theorems of Levi [13] and Grunbaum [10], and the separation theorem (*Mazur Lemma*, see theorem (T6) below) derived by Ellis ([8]) under general assumptions. Prenowitz and Jantosciak [17] and Bryant and Webster ([3], [4], [5]) derived many convexity theorems for *Join Geometries*, where the primitives are *joins* (i.e. intervals joining pairs of points.) These works are close in spirit to ours, but our axiomatic approach is different. Indeed, the choice of *affine sets* as primitives , analogous to the classical (Hilbert) approach to Euclidean geometry in two and three dimensions, enables the Ordered Incidence Geometry to develop along familiar classical lines, resulting in economy and elegance as well as a clear understading of the roles of specific classical axioms in securing convexity theorems.

Most of the above mentioned axiomatic settings suffer from a lack of *concrete models*, in particular, models leading to useful applications in a manner

similar to the way separation theorems are used to derive duality relations in optimization theory. An effort is made here to present concrete models . see Section 3 below. The applicability of our theory is demonstrated in Section 8 where the separation theorem (T7) is used to obtain a *Fenchel-type duality theorem* for *sub-F functions*. (see [1] and [18], Sections 84-85.) The graphs of sub-F functions are the (non-vertical) lines in a geometry, (see model (M6) in Section 3), of which the above mentioned Moulton Geometry is a special case.

The paper has nine sections. The contents are outlined below:

1. Axioms. Ordered incidence geometry. Axioms of incidence, **(A1)-(A5)**, and order, **(A6)- (A9)**, for affine sets. Affine hulls, **(D1)**.

2. Immediate consequences. Basic properties of affine sets, **(C1)-(C8)**, and separation, **(D2)** and **(T1)**.

3. Models. Concrete models, **(M1)-(M7)**, of ordered incidence geometries. Beckenbach geometries. Generating new geometries from old.

4. Triangles. Basic lemmas on triangles, **(L2)-(L5)**, needed in the sequel.

5. Lineal hulls. Characterization of affine sets in terms of lineal hulls, **(T2)**.

6. Convex sets. Definitions **(D5)-(D7)** and basic properties **(L7)**, **(T3)-(T4)** of convex sets, their relative cores and closures. Simplices and dimension, **(D8)-(D9)**, **(L8)- (L9)**. Finite-dimensional convex sets have nonempty relative cores, **(T5)**.

7. Separation. Halfspaces corresponding to a hyperplane, **(L10)-(L11)** and **(D11)**. Separation, **(D12)** and **(L12)**. Convex pairs, **(D13)** and **(T6)**. The completeness axiom **(A10)**. Complete ordered incidence geometry, **(D15)** and **(L13)**. The hyperplane corresponding to a maximal convex pair, **(L14)**, and the separation theorem **(T7)**.

8. Applications to functions on the real line. Sub-F and super-F functions,

(D16). Epigraphs, (L15) and (L16). Conjugates, (D17) and (L17). A Fenchel duality theorem, (T8).

9. The theorems of Radon and Helly. **Radon's theorem (T9) and Helly's theorem (T10) follow by a standard argument, (L18).**

1. AXIOMS

An *Ordered Incidence Geometry* (abbreviated *OIG*) , $\underline{\mathbf{G}}$, is a triple

$$\underline{\mathbf{G}} = \{\mathbf{X}, \underline{\mathbf{A}}, dim\}$$

and an order relation *(betweeness)*, endowed with nine axioms (A1)–(A9) given below. Here

X is the *space* of elements *(points)*,

$\underline{\mathbf{A}}$ is a family of subsets of X, called the *affine sets* of $\underline{\mathbf{G}}$,

***dim* is an integer valued function on $\underline{\mathbf{A}}$, called the *dimension*.**

An affine set $A \in \underline{\mathbf{A}}$ is called a *k-affine* if $dim\ A = k$. In particular, we use the terms,

***point* for a 0-affine [1]**

***line* for a 1-affine,**

***plane* for a 2-affine, and**

***hyperplane* for a $(dim\mathbf{X} - 1)$ - affine.**

By convention $dim\ \emptyset = -1$, where $\emptyset$ is the *empty set*.

(A1) Axiom. **$\underline{\mathbf{A}}$ contains X, the empty set $\emptyset$ and all singletons $\{ x \}$, $x \in \mathbf{X}$.**

(A2) Intersection Axiom. **$A, B \in \underline{\mathbf{A}}$ implies $A \bigcap B \in \underline{\mathbf{A}}$.**

(D1) Definition. **For $S \subset \mathbf{X}$ we define the *affine hull* of S, $\mathbf{a}(S)$, by**

$$\mathbf{a}(S) = \bigcap \{ A : A \in \underline{\mathbf{A}}, S \subset A\}$$

which by (A2) is an affine set.

---

[1] It should be clear from the context whether a "point" refers to an element of X or of $\underline{\mathbf{A}}$.

The following three axioms (A3)–(A5) express *monotonicity properties* of the dimension function.

(A3) Axiom. For $A, B \in \underline{\mathbf{A}}$. $A \subset B$ implies $dim\ A \leq dim\ B$.

(A4) Axiom. For $x \in \mathbf{X}$, $A \in \underline{\mathbf{A}}$.

$$x \notin A \quad implies \quad dim\ \mathbf{a}(A \bigcup x) = dim\ A + 1$$

(A5) Axiom. For $A, B, H \in \underline{\mathbf{A}}$.

$$\text{if } \begin{cases} B, H \subset A, & dim\ H = dim\ A - 1 \\ B \bigcap H \neq \emptyset, & B \text{ not contained in } H \end{cases}$$

$$then \quad dim\ B \bigcap H = dim\ B - 1.$$

The remaining axioms (A6)–(A9) define the order relation *betweeness*. Points lying on the same line are called *collinear*. The above axioms imply that two distinct points a, b determine a unique line $\overline{ab}$ containing them, see Corollary (C6). In fact $\overline{ab} = \mathbf{a}(a, b)$.

For distinct collinear points a, b, c we denote by

$$abc$$

the fact that b is between a and c. The set of all points between a and b is the *open segment* or *open interval* joining a, b denoted by

$$(a, b)$$

(A6) Axiom. abc is equivalent to cba.

(A7) Axiom If $a \neq c$ then there exist points b, d such that

$$abc \quad \text{and} \quad acd.$$

(A8) Axiom. If a, b, c are distinct and collinear then one and only one of them is between the other two.

(A9) The Pasch Axiom. If a, b, c are non-collinear, and if L is a line in

$\mathbf{a}(a, b, c)$ with

a, b, c not in L,

$L \cap (a, b) \neq \emptyset$

then either

$L \cap (a, c) \neq \emptyset$,

or

$L \cap (b, c) \neq \emptyset$.

2. IMMEDIATE CONSEQUENCES

The results in this section come close to the axioms, so close in fact that they could be stated as alternative axioms.

We denote by #(S) the *number of elements* of the set S.

An easy consequence of definition (D1) is the following,

(C1) Corollary. For any affine set A,

$$S \subset A \text{ implies } \mathbf{a}(S) \subset A.$$

□

For $\#S = 2$, $dim\ A = 2$ this is Hilbert's Incidence Axiom I,6 [11].

(C2) Corollary. If A is a k-affine, $A \neq \emptyset$, then there is a subset S of A with

$$\#(S) = k+1$$

and

$$dim\ \mathbf{a}(S) > k-1.$$

Proof. Let $x_1 \in A$, $S_1 = \{x_1\}$.

If $k > 0$ then $A \neq \mathbf{a}(S_1)$. Therefore there is an $x_2 \in A$, $x_2 \notin \mathbf{a}(S_1)$.

Let $S_2 = S_1 \bigcup \{x_2\}$.

If $k > 1$ then $A \neq \mathbf{a}(S_2)$ and there exists an $x_3 \in A$. $x_3 \notin \mathbf{a}(S_2)$.
Let $S_3 = S_2 \bigcup \{x_3\}$, etc.

Repeating this k times gives the claimed set S.

□

For $k = 1, 2, 3$ this corollary reduces to Hilbert's Incidence Axioms I,3 and I,8 specifying the existence of:

(i) two distinct points on any line,

(ii) three non-collinear points on any plane, and

(iii) four non-coplanar points in the 3-dimensional space, see [11].

(C3) Corollary. If $S \subset \mathbf{X}$, $x \in \mathbf{X}$, then

$$\mathbf{a}(S \bigcup x) = \mathbf{a}(\mathbf{a}(S) \bigcup x)$$

Proof. Follows from definition (D1).

□

(C4) Corollary. If $S \subset \mathbf{X}$, $\#(S) = k+1$, then $dim\ \mathbf{a}(S) \leq k$.

Proof. Let $S = \{x_1, x_2, \cdots x_{k+1}\}$,

and for $i = 1, \cdots, k+1$ let $S_i = \{x_1, \cdots, x_i\}$.

Then

$$dim\ \mathbf{a}(S_1) = 0,$$

and for $i = 2, \ldots, k+1$,

$$dim\ \mathbf{a}(S_i) = dim\ \mathbf{a}(\mathbf{a}(S_{i-1} \bigcup x_i), \quad by\text{(C3)}$$

$$= \begin{cases} dim\ \mathbf{a}(S_{i-1}) & \text{if } x_i \in \mathbf{a}(S_{i-1}) \\ dim\ \mathbf{a}(S_{i-1}) + 1 & otherwise,\ by \text{ (A4)} \end{cases}$$

□

(C5) Corollary. Let A, B be k-affines, $S \subset A \bigcap B$. Then either

$$\mathbf{a}(S) = A = B$$

or

$$dim\ \mathbf{a}(S) < k.$$

Proof. From $S \subset A \cap B$ follows $\mathbf{a}(S) \subset A \cap B$, by (D1), (A2).
If $dim\ \mathbf{a}(S) = k$ then,

$$\mathbf{a}(S) \subset A \cap B \subset A \text{ or } B \qquad (2.1)$$

shows by (A3) that

$$dim\ A \cap B = k$$

in which case equalities hold in (2.1) by (A4).

□

A similar consequence of (A3), (A4) is

(C6) **Corollary.** Let $A, B \in \underline{\mathbf{A}}$, $A \subset B$. Then

$$A = B \quad <=> \quad dim\ A = dim\ B$$

□

The following converse of (C2) can now be proved:

(C7) **Corolary.** If $S \subset \mathbf{X}$, $\#S = k+1$, $dim\ \mathbf{a}(S) > k-1$ then there is a unique k-affine A containing S.

Proof. Follows from (C4) and (C5)

□

For $k = 1$ and 2, this corollary reduces to Hilbert's Incidence Axioms I,1-2 and I,4-5, respectively, [11].

The following corollary states roughly that if an affine set $\mathbf{a}(S)$ is "over-determined" by S, then certain points of S are "affine combinations" of others.

(C8) **Corollary.** If $S \subset \mathbf{X}$, $\#S = k+1$, $dim\ \mathbf{a}(S) \leq k-1$ then there is an $x \in S$ such that $x \in \mathbf{a}(S\ /\ x)$.

Proof. Otherwise, by (**A4**),

$$dim\ \mathbf{a}(S) = dim\ \mathbf{a}\{S \ / \ x\} + 1, \text{ for all } x \in S.$$

$$= dim\ \mathbf{a}(S \ / \ x \bigcup y) + 2, \text{ for all } x,\ y \in S,$$

, etc.

□

In plane geometry it is well known that the Pasch Axiom (A9) is equivalent to a *Plane Separation Axiom* , ([14], Chapter 12). This equivalence holds also in our geometry. First we require the following:

(**D2**) **Definition.** Let A, H be affine sets, $A \subset H$. Then H *separates* A if for any two points $x,\ y \in A \ / \ H$ such that,

$$(x,\ y) \bigcap H \neq \emptyset \tag{2.2}$$

there is no point $z \in A \ / \ H$ such that,

$$(x,\ z) \bigcap H = \emptyset \quad and \quad (y,\ z) \bigcap H = \emptyset \tag{2.3}$$

i.e. x and y cannot be, at the same time, on "opposite" sides of H and on the "same" side of H.

We have now the following

(**T1**) **Theorem.** Let A, H be affine sets, $dim\ A \geq 1$, $H \subset A$, $dim\ H = dim\ A - 1$. Then H separates A.

Proof. The case $dim\ A = 1$ follows from the order axiom (A8).

Let $dim\ A \geq 2$, and suppose H does not separate A, i.e. there are distinct points $x,\ y,\ z \in A \ / \ H$ satisfying (2.2) and (2.3). It follows from (A8) that $x,\ y,\ z$ are non-collinear. Let P be the plane through $x,\ y,\ z$. The intersection $P \bigcap H$ is

(i) H if $dim\ A = 2$ (i.e. $P = A$),

(ii) a line if $dim\ A > 2$ (by (A5)),

so that, in either case, $P \cap H$ is a line, say L.
Since L intersects (x, y), it follows from (A9) that L also intersects (x, z) or (y, z), violating (2.3).

□

Conversely, if (T1) is postulated then (A9) easily follows, i.e. the two are equivalent.

3. MODELS

In this section we give concrete models of Ordered Incidence Geometries.

(M1) **The real Euclidean n-dimensional space.** Here $\mathbf{X} = R^n$, and $\underline{\mathbf{A}}$ and *dim* agree with their standard vector space meanings. Specifically, A is a k-affine if and only if

$$A = \{x : x = Mu + b,\ u \in R^m\}$$

where M is an $n \times m$ matrix of rank k.

(M2) **Convex restriction of R^n.** Let $\mathbf{X}$ be a given open convex subset of R^n, and let $\underline{\mathbf{A}}$ consist of the restrictions of the affine sets of (M1) to $\mathbf{X}$.

For $n = 2, 3$, this model includes the *Cayley-Klein Incidence Plane* and *Space*, respectively , [14], Chapter 5. We note that (M2) is a non-Euclidean geometry, since through a point x off a line L there pass more than one parallel to the line, see Fig 3.1.

The following four models represent 2-dimensional geometries, with $\mathbf{X}$ a subset of the *(x, y) - plane*. In each case the 0-affines are the ordinary points, and the 2-affine is $\mathbf{X}$. *Betweeness* is to be understood in the natural way.

(M3) **The Poincare Incidence Plane.** Here,

$\mathbf{X} = \{(x, y): x^2 + y^2 < 1\}$, the interior of the unit circle,

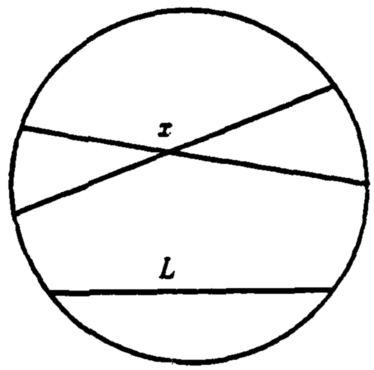

Fig 3.1

$U = \{(x, y): x^2 + y^2 = 1\}$. 1-affines are the restrictions to $\mathbf{X}$ of:

(i) lines through the origin (0, 0), and

(ii) circles which intersect U at right angles.

See Fig 3.2 where ab, cd are lines.

(M4) The Poincare Half-Plane Incidence Plane.

$\mathbf{X} = \{(x, y): y > 0\}$, the upper half-plane,

1-affines are the restrictions to $\mathbf{X}$ of

(i) vertical lines, and

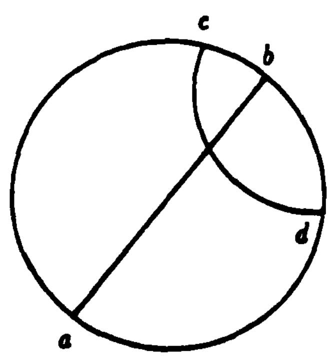

Fig 3.2

(ii) circles with centers on the x-axis.

(M5) The Moulton Incidence Plane.

X = the (x, y)-plane.

1-affines are of three types:

(i) vertical lines,

(ii) lines with nonpositive slope, and

(iii) bent lines, given for any $m > 0$, by

$$y = \begin{cases} mx + \beta & , \quad \text{if } x \leq 0 \\ \frac{m}{2}x + \beta & , \quad \text{if } x > 0 \end{cases}$$

The last two models are special cases of what we call *Beckenbach Geometries.*

First we require

(D3) Definition. Let (a, b) be an open interval in R. A family **F** of functions $F:(a, b) \to R$ is a *Beckenbach Family* (a *B - Family* for short) if:

(i) each $F \in \mathbf{F}$ is continuous on (a, b).

(ii) For any two points (x_1, y_1), (x_2, y_2) with

$$a < x_1 < x_2 < b$$

there exists a unique $F \in \mathbf{F}$, denoted by F_{12}, such that

$$F_{12}(x_i) = y_i, \quad (i = 1,2) \tag{3.1}$$

Such families were introduced by Beckenbach in [1], see also ([18], Sections 84-85.)

(M6) Beckenbach Geometries. Let

(a, b), (c, d) be open intervals (not necessarily bounded), and let

F be a B - family of functions $F:(a, b) \to (c, d)$,

A *Beckenbach Geometry* is a two dimensional geometry with

$\mathbf{X}$ the Cartesian product $(a, b)\times(c, d)$.

and the 1-affine through any pair of points $(x_1, y_1), (x_2, y_2)$ in $\mathbf{X}$ is.

(i) the vertical line $x = x_1$ if $x_1 = x_2$,

(ii) the graph of F_{12} (defined by (3.1)) if $x_1 \neq x_2$.

A Beckenbach Geometry (henceforth *B - geometry*) is completely determined by the B - family **F**. To underscore this dependence we use the notation

$$\underline{\mathbf{G}_{\mathbf{F}}}$$

For the B - geometry corresponding to **F**.

For a B - geometry, axioms (A1) - (A8) are easily shown to hold. The validity of axiom (A9) can be verified using the following result of Beckenbach ([1], Theorem 1), which is of independent interest.

(L1) **Lemma.** Let $a < x_0 < b$ and let F_α, F_β be two distinct members of **F** such that

$$F_\alpha(x_0) = F_\beta(x_0),$$

then

$$F_\alpha(x) > F_\beta(x), \text{ for all } x \text{ in } (a, b) \text{ on one side of } x_0,$$

$$F_\alpha(x) < F_\beta(x), \text{ for all } x \text{ in } (a, b) \text{ on the other side of } x_0.$$

□

Examples of B - families

In the following examples the B - families are given in a parametric form,

$$\mathbf{F} = \{F : F(x) = F(x; \alpha, \beta)\}$$

which we abbreviate by,

$$\mathbf{F} = \{F(x; \alpha, \beta\}$$

(E1) **The affine functions.** Here $(a, b) = R = (c, d)$,

$$\mathbf{F}_1 = \{\alpha x + \beta : \alpha, \beta \in R\}$$

The resulting geometry $\underline{\mathbf{G}}_{\mathbf{F}_1}$ is the Euclidean plane geometry.

(E2) $(a, b) = R, \ (c, d) = (0, \infty)$.

$$\mathbf{F}_2 = \{ +\sqrt{\beta - (x-\alpha)^2} : \beta > 0, \alpha \in R\}$$

Here $\underline{\mathbf{G}}_{\mathbf{F}_2} <=>$ (M.4), the Poincare half-plane incidence plane.

(E3) **Bent lines.** Let $(a, b) = R = (c, d)$ and let $\mathbf{F}_3$ be the family of functions whose graphs are the 1-affines of model (M5). Then

$\underline{\mathbf{G}}_{\mathbf{F}_3} <=>$ (M5), the Moulton Incidence Plane.

(E4) $\mathbf{F}_4(x; \alpha, \beta) = \{\alpha\phi_1(x) + \beta\phi_2(x) + \phi_3(x)\}$

ϕ_i is continuous, $i = 1, 2, 3$

$\phi_2(x) > 0$ on (a, b),

$\alpha, \beta \in R$.

A necessary and sufficient condition for $\mathbf{F}_4$ to be a B-family is that

$$\frac{\phi_1}{\phi_2}$$

is a strictly monotone function. Thus, for example, for $(a, b) = R$,

$\mathbf{F} = \{\alpha e^x + \beta e^{-x}\}$ is a B-family,

while

$\mathbf{F} = \{\alpha x^2 + \beta\}$ is not.

(E5) $\mathbf{F}_5(x; \alpha, \beta) = \phi(\alpha, x) - \beta$

ϕ differentiable in α for all x.

Here a necessary and sufficient condition for $\mathbf{F}_5$ to be a B-family is that

$$\frac{\partial\phi}{\partial\alpha}$$

is a strictly monotone function of x.

(E6) $\mathbf{F}_6(x; \alpha, \beta) = a(\alpha)u(x) + b(\alpha)v(x) - \beta$,

where

where

a and u are strictly increasing.

b and v are strictly decreasing.

For example.

$$\mathbf{F} = \{\cosh(\alpha + x) - \beta : \alpha, \beta \in R\}$$

Generating new geometries from old

Given a geometry $\underline{\mathbf{G}} = \{\mathbf{X}, \underline{\mathbf{A}}, dim\}$, there are three obvious rules for generating new geometries from $\underline{\mathbf{G}}$.

Rule 1. Let A_0 be a fixed affine set, and let $\underline{\mathbf{G}}_{A_0}$ be the *restriction* of $\underline{\mathbf{G}}$ to A_0,

$$\underline{\mathbf{G}}_{A_0} = \{A_0, A_0 \cap \underline{\mathbf{A}}, dim\}$$

where

$$A_0 \cap \underline{\mathbf{A}} = \{A_0 \cap A : A \in \underline{\mathbf{A}}\}$$

is the collection of affine sets (with dimensions $\leq dim\ A_0$) of $\underline{\mathbf{G}}_{A_0}$.

Rule 2. Let C be an "open convex" subset of X, and let $\underline{\mathbf{G}}_C$ be the restriction of $\underline{\mathbf{G}}$ to C

$$\underline{\mathbf{G}}_C = \{C, C \cap \underline{\mathbf{A}}, dim\}.$$

This rule was used in getting (M2) from (M1). The meaning of "openness" and "convexity" (with respect to $\underline{\mathbf{G}}$) will be specified in Section 6.

Rule 3. Let $H : \mathbf{X} \to \mathbf{Y}$ be one-to-one and onto. From $\underline{\mathbf{G}} = \{\mathbf{X}, \underline{\mathbf{A}}, dim\}$ we get the geometry

$$\underline{\mathbf{G}}_H = \{\mathbf{Y}, H(\underline{\mathbf{A}}), \underline{dim}\}$$

where $H(\underline{\mathbf{A}}) = \{H(A) : A \in \underline{\mathbf{A}}\}$,

and $\underline{dim}\ H(A) = dim\ A$, for all $A \in \underline{\mathbf{A}}$

For example, (M4) is obtained from (M3) by the transformation,

$$H(x,\ y) = \begin{pmatrix} \dfrac{2y}{(1+x)^2+y^2} \\ \dfrac{1-x^2-y^2}{(1+x)^2+y^2} \end{pmatrix}$$

which maps the unit circle U onto the x-axis, and its interior onto the open upper half-plane.

Although $\underline{G}$ and $\underline{G}_H$ are isomorphic, one of them may prove to be more convenient than the other. For example, it is easier to verify the Pasch Axiom (A9) in (M4) than in (M3).

We end this section with a 3-dimensional OIG, a generalization of (M4).

(M7) The 3-Dimensional Poincare Half-Space. Here,

$X = \{\ (x,\ y,\ z) : z > 0\ \}$ the open upper half-space,

2-affines are the restrictions to X of

(i) vertical planes (i.e. perpendicular to the (x, y)-plane), or

(ii) half-spheres with center in the (x, y)-plane.

1-affines are the intersections of (different and intersecting) 2-affines, i.e.

(i) vertical lines, or

(ii) half-circles with center in the (x, y)-plane, contained in vertical planes.

For example, the plane $a(p_1,\ p_2,\ p_3)$ through three given non-collinear points

$$p_i = (x_i,\ y_i,\ z_i),\quad i = 1,\ 2,\ 3$$

is constructed as follows:

(a) Let E be the Euclidean plane through p_1, p_2, p_3.

(b) In E, find the intersection q of the three (or of any two) perpendicular bisectors of the triangle $\Delta p_1\ p_2\ p_3$.

(c) Let L be the line, perpendicular to E, through q.

(d) If L does not intersect the (x, y)-plane, then E is a vertical plane and

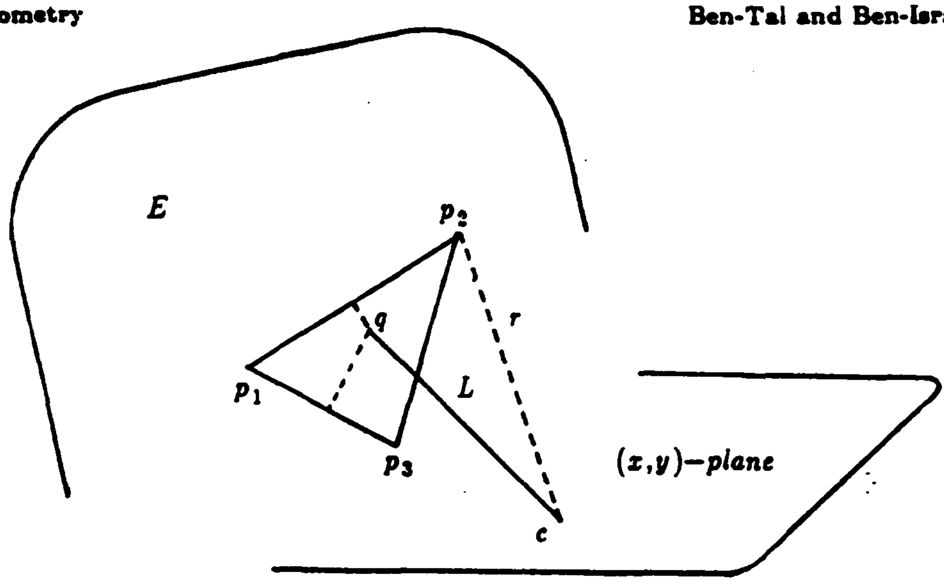

Fig 3.3 Construction of $\mathbf{a}(p_1, p_2, p_3)$

its restriction to X is the sought $\mathbf{a}(p_1, p_2, p_3)$.

(e) Otherwise, let c be the point where L intersects the (x, y)-plane. Then $\mathbf{a}(p_1, p_2, p_3)$ is the restriction to X of the sphere with center at c, and radius r equal to the length of the segment cp_i, (any i will do.) .

It is easy to check that (M7) satisfies (A1) - (A8). To verify (A9), use its equivalent form (T1). In particular, the two sides of a (nonvertical) 2-affine

$$P = \{(x, y, z): z > 0, (x - \alpha)^2 + (y - \beta)^2 + z^2 = r^2\}$$

are the "inside" of P,

$$\{(x, y, z): z > 0, (x - \alpha)^2 + (y - \beta)^2 + z^2 < r^2\}$$

and its "outside".

4. TRIANGLES

Any three non-collinear points $\{a, b, c\}$ constitute a *triangle* Δabc, taken here to mean an abstract geometrical object.

Basic properties of triangles in an OIG are given in lemmas (L2) -

(L5). stated for a general triangle Δabc. These lemmas are used repeatedly in the sequel.

(**L2**) **Lemma.** For any $u \in (a, c)$ and $v \in (u, b)$, there is a point $w \in (b, c)$ such that.

$$v \in (a, w).$$

Proof.

Applying (A9) to the triangle Δubc, it follows that the line $\overline{av}$ intersects either (b, c) or (u, c). Let w be that intersection point.

If $w \in (u, c)$ then the lines $\overline{av}$ and $\overline{ac}$ coincide, by (C5). Therefore v is on the line $\overline{ac}$, which (by (C5) again) coincides with the line $\overline{bu}$. This shows $\{a, b, c\}$ to be collinear, a contradiction.

$\square$

The following is a sort of converse of (L2).

(**L3**) **Lemma.** For any $u \in (a, c)$ and $w \in (b, c)$ there is a point v in the intersection

$$(a, w) \cap (b, u).$$

Proof. Applying (A9) *to* Δubc, *it follows that* (a, w) intersects either (b, u) or (u, c). The latter is impossible, for it implies that the lines $\overline{aw}$ and $\overline{ac}$ coincide,

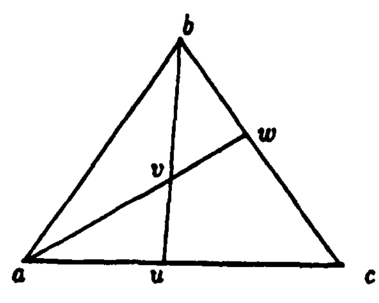

Fig 4.1 Illustration of (L2) *and* (L3)

and thus that $\{a, b, c\}$ are collinear.

□

(L4) Lemma. Let $u \in (a, c)$, $w \in (b, c)$. Then for any $v \in (u, w)$ there is a $z \in (a, b)$ such that

$$v \in (c, z).$$

Proof. By (L2) applied to Δbuc there is a $w_1 \in (b, u)$ such that $v \in (c, w_1)$. Applying (L2) again to Δabc shows the existence of a $z \in (a, b)$ such that

$$w_1 \in (c, z)$$

$$\therefore v \in (w_1, c) \subset (c, z)$$

□

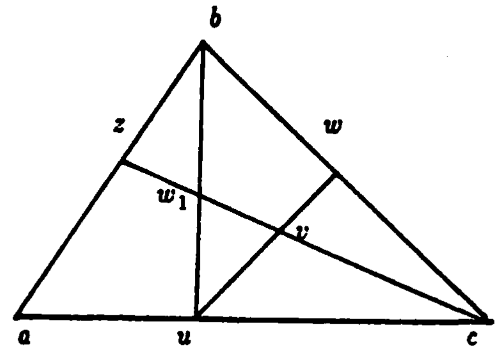

Fig. 4.2 Illustration of (L4)

Again, a sort of converse of (L4) can be given:

(L5) Lemma. Let $z \in (a, b)$. Then for any $v \in (c, z)$ there exist two points

$$u \in (a, c) \quad , \quad w \in (b, c)$$

such that

$$v \in (u, w).$$

Proof. Applying (L2) to Δabc shows that there is a $v_1 \in (a, c)$ such that $v \in (b, v_1)$.

Let $u \in (a, v_1)$. By (A9) (applied to Δbv_1c) it follows that the line $\overline{uv}$

intersects either $(b.\ c)$ or $(v_1,\ c)$. The latter leads to contradiction. so let w be the point where $\overline{uv}$ intersects $(b.\ c)$. Then

$$v \in (u.\ w)$$

□

5. LINEAL HULLS

In the Euclidean geometry R^n a set A is *affine* if and only if

$$A = \{\sum \lambda_i x_i : x_i \in A\,,\ \sum \lambda_i = 1\ \} \tag{5.1}$$

i.e. A coincides with the set of *affine combinations* of its elements. The analogous geometrical representation in an OIG (where algebraic constructions such as (5.1) are not available) is given in (T2). First we require:

(D4) Definition. For a given subset S of $\mathbf{X}$, the *lineal hull of S*, $\mathrm{l}(S)$, is

$$\mathrm{l}(S) = \bigcup\{\overline{xy} : x,\ y \in S\ \}$$

the union of lines through pairs of points in S.

For a point x, we define

$$\mathrm{l}(x) = \{x\}.$$

We also use the abbreviation

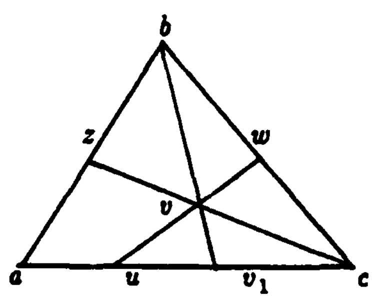

Fig 4.9 Illustration of (L5)

$$\mathbf{l}^{(2)} = \mathbf{l}(\mathbf{l}(S)).$$

(**L6**) **Lemma.** If A is a k-affine, $x \notin A$, then

$$\mathbf{a}(A \bigcup x) = \mathbf{l}^{(2)}(A \bigcup x)$$

Proof.

<u>$\mathbf{l}^{(2)}(A \bigcup x) \subset \mathbf{a}(A \bigcup x)$:</u> Follows by applying (C1) twice.

<u>$\mathbf{l}^{(2)}(A \bigcup x) \supset \mathbf{a}(A \bigcup x)$:</u> Let z be any point in $\mathbf{a}(A \bigcup x)$, and consider two cases:

(i) $(x, z) \bigcap A \neq \emptyset$.

Let $u \in (x, z) \bigcap A$.

$$\therefore \quad z \in \overline{ux} \subset \mathbf{l}(A \bigcup x)$$
$$\subset \mathbf{l}^{(2)}(A \bigcup x)$$

(ii) $(x, z) \bigcap A = \emptyset$.

Let u be any point in A. Then, by (A7), the line $\overline{ux}$ contains a point y such that

$$u \in (y, x)$$

Since $(x, y) \bigcap A \neq \emptyset$, $(x, z) \bigcap A = \emptyset$, it follows from (2.3) that

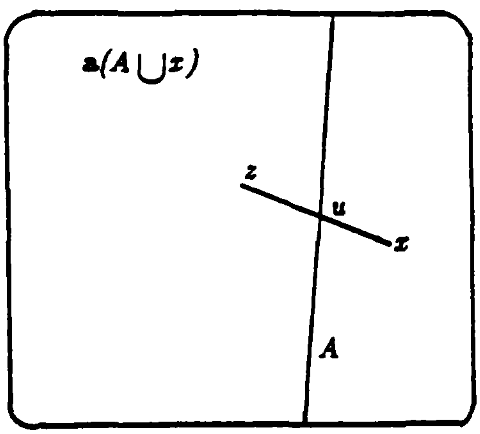

Fig 5.1 Case (i)

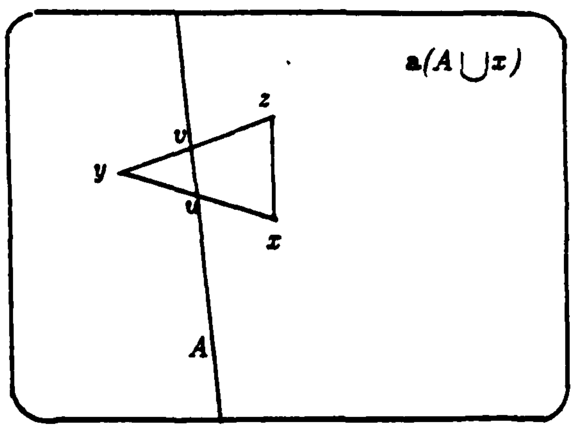

Fig 5.2 Case (ii)

$$(y, z) \cap A \neq \emptyset$$

Let $v \in (y, z) \cap A$. Then

$$z \in l(A \cup y) \subset l^{(2)}(A \cup x).$$

□

(T2) Theorem. S is an affine set if and only if

$$S = l(S) \tag{5.2}$$

Proof. Only if. $S \subset l(S)$ always holds, and $l(S) \subset S$ follows from (C1).

If. Let A be a maximal affine set contained in S. If $A = S$ then S is affine. Otherwise, let $x \in S / A$.

$\therefore \quad S \supset A \cup x$

$\therefore \quad S = l(S) \supset l(A \cup x)$, by (5.2).

$\therefore \quad S = l^{(2)}(S) \supset l^{(2)}(A \cup x) = \mathbf{a}\,(A \cup x)$, by (L6),

contradicting the maximality of A, since $\mathbf{a}(A \cup x)$ strictly contains A.

□

6. CONVEX SETS

The basic properties of convex sets are developed in this section.

(D5) Definitions. A set $S \subset \mathbf{X}$ is:

(i) ***star shaped*** at x if for all $y \in S$,

$$(x, y) \subset S,$$

(ii) ***convex*** if for any two points $x, y \in S$,

$$(x, y) \subset S.$$

(D6) Definition. For any set $S \subset \mathbf{X}$, the ***convex hull of*** S, $\mathrm{conv}(S)$, is the intersection of all convex sets containing S.

(D7) Definitions. For any set $S \subset \mathbf{X}$,

(i) the ***core of*** S, core S, is

$$\textbf{core } S = \{z \in S : \forall y \in \mathbf{X},\ y \neq x,\ \exists\, z \in (x, y) \textit{ such that } (x, z) \subset S\ \} \qquad (6.1)$$

(ii) the *relative core of S*, **relcore** S, is defined by (6.1) with

$$\forall y \in \mathbf{a}(S) \text{ replacing } \forall y \in \mathbf{X}.$$

(iii) the *set linearly accessible from S*, **lina** S, is

$$\textbf{lina } S = \{y \in \mathbf{X} : \exists x \in S \textit{ such that } (x, y) \subset S\ \}.$$

(iv) the *closure of S*, **cl** S, is

$$\textbf{cl } S = S \bigcup \textbf{lina } S.$$

(**L7**) **Lemma.** Let a set S with a nonempty relative core [2] be star shaped at p. Then

$$x \in \textbf{relcore } S \qquad \text{implies} \qquad (p, x) \subset \textbf{relcore } S,$$

i.e. **relcore** S is also star shaped at p.

Proof. Let y be any point in $\mathbf{a}(S)$, and distinguish two cases:

(i) $y \in \overline{px}$.

From $(p, x) \subset S$ (since S is star shaped),

$(x, z) \subset S$ for some $z \in (x, y)$, (since $x \in$ **relcore** S),

it follows for any $u \in (p, x)$ that

$$(u, v) \in S \quad \text{for some} \quad v \in (u, y).$$

(ii) $y \notin \overline{px}$, (see Fig. 6.1)

Since $x \in$ **relcore** S there is a $v \in (y, x)$ such that

$$(x, v) \subset S \qquad (6.2)$$

Let x_1 be any point in (p, x). Lemma (L3), applied to Δypx, implies that (x_1, y) and (p, v) intersect at a point, say z.

The proof is completed by showing that $(x_1, z) \in S$. Indeed let $x_2 \in (x_1, z)$. Then Lemma (L4), applied to Δxpv, shows that there is a point $u \in (v, x)$ with $x_2 \in (p, u)$.

[2] In the finite-dimensional case, **relcore** $S \neq \emptyset$ for any convex set S, see (**T5**).

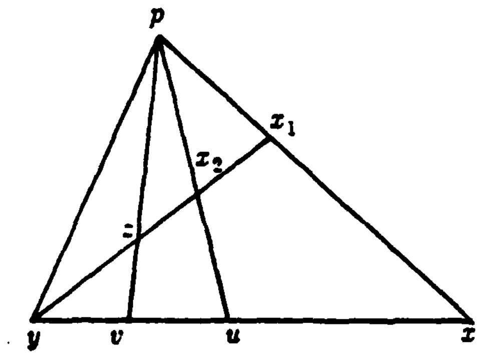

Fig 6.1

But $u \in S$, by (6.2).

$\therefore \qquad x_2 \in (p,\, u) \subset S$ since S is star shaped.

□

(T3) Theorem. Let S be a convex set, $y \in$**lina** S, $x \in$**relcore** S, $y \neq x$. Then

$$(x,\, y) \subset \textbf{relcore}\ S.$$

Proof. Let $x_1 \in (x,\, y)$. We show first that:

$\underline{x_1 \in S}$. Since $y \in$**lina** S, there is a $y \neq z \in S$ such that

$$(z,\, y) \subset S$$

There are two cases:

(i) $z \in \overline{yx}$.

Here either $z \in (y,\, x_1)$,

or $y \in (z,\, x_1)$

or $x_1 \in (z,\, y)$

and in each case, $x_1 \in S$, by Axiom (**A7**) and the convexity of S.

(ii) $z \notin \overline{yx}$, (see Fig. 6.2)

Let $w \in \overline{zy}$ be such that $y \in (z,\, w)$. By Lemma (**L2**), applied to Δzwx. there is a By Lemma (**L2**), applied to Δzwx, there is a $z_1 \in (x,\, w)$ such that $x_1 \in (z,\, z_1)$.

Since **lina** $S \subset$ **a**(S), by (A7), it follows, in that order. that

$$y, \; w \text{ and } z_1$$

are in **a**(S).

Since $x \in$ **relcore** S, there is then a $\overline{z_2} \in (x, z_1)$ such that

$$(x, \overline{z_2}) \subset S.$$

Let $z_2 \in (x, \overline{z_2})$. Applying Axiom (A9) to $\Delta z z_1 w$, and then to Δxyz, it follows that the line $\overline{z_2 x_1}$ intersects (z, w) at a point $z_3 \in (y, z)$.

From $z_3 \in (y, z) \subset S$,

$z_2 \in (x, \overline{z_2}) \subset S$, and the convexity of S it follows that

$$x_1 \in S.$$

We complete the proof by noting that, for any $x_2 \in (x_1, y)$, it is also true that $x_2 \in (x, y)$, and (as proved above for x_1),

$$x_2 \in S.$$

Since S is convex, it is star shaped at x_2, and by Lemma (L7),

$$(x, x_2) \subset \textbf{relcore } S$$

showing that

$$x_1 \in \textbf{relcore } S.$$

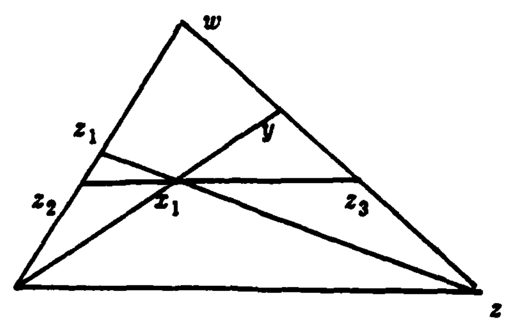

Fig 6.2

□

Remark. The assumption

$$x \in \textbf{relcore } S$$

cannot be omitted in (T3). Consider for example Fig 6.3, where the convex set S is missing the upper side except for x. Here $x \in S$, $y \in$ **lina** S but (x, y) is not contained in S.

(**T4**) **Theorem.** Let S be a convex set. Then,

(a) **relcore** S,

(b) **cl** S,

are convex.

Proof.

(a) Follows from (L7) since a convex set is star shaped at each of its points.

(b) Let $x, y \in$ **cl** S ($= S \bigcup$ **lina** S.) We show that $(x, y) \subset$ **cl** S. Only two cases require proof:

(i) $x \in S$, $y \in$ **lina** S.

Let $z \in S$ be such that $(z, y) \subset S$, (see (D7), (iii)).

For any $w \in (x, y)$ we show that $w \in$ **lina** S proving that

$$(x, y) \subset \textbf{cl } S$$

Indeed let $u \in (w, z)$, see Fig. 6.4. Then, by (L2), the line $\overline{xu}$ intersects (y, z)

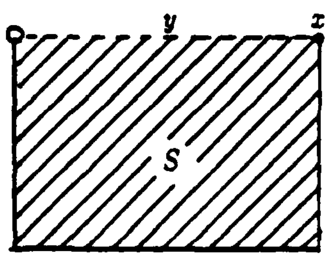

Fig 6.3

at some point v. Since $x \in S$, $v \in S$ it follows that $u \in S$, and so $(w, z) \subset S$.

(ii) $x, y \in$ **lina** S

Let z_1, z_2 be such that

$$(z_1, x) \subset S, \ (z_2, y) \subset S.$$

We may assume here that $z_1, z_2 \in S$.

Let $w \in (x, y)$. We again prove that $w \in$ **lina** S, by showing that (see Fig. 6.5)

$$(z_1, w) \subset S.$$

Indeed let $u \in (z_1, w)$. Then, by (L5) applied to Δxyz_1, there exist two points $u_1 \in (x, z_1)$ and $u_2 \in (x, z_2)$ such that

$$u \in (u_1, u_2).$$

From Axiom (A9) it follows then that the line $\overline{u_1 u_2}$ intersects either (y, z_2) or (z_1, z_2) at a point u_3. In either case $u_3 \in S$, and therefore

$$u \in (u_1, u_3) \subset S.$$

□

We end this section with a result of a topological nature, Theorem (T5), stating that a (nonempty) finite-dimensional convex set has a nonempty relative core.

(D8) Definition. For any $S \subset \mathbf{X}$, the *dimension* of S, *dim* S, is

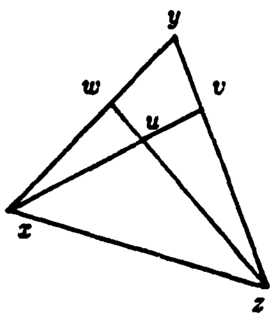

Fig. 6.4

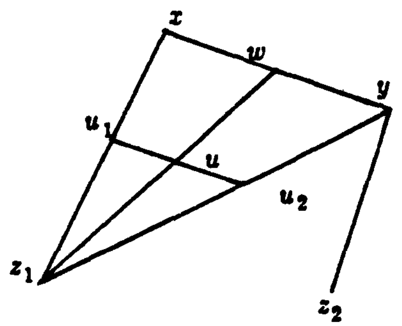

Fig. 6.5

$$dim\ S = dim\ \mathbf{a}(S)$$

the dimension of the affine hull of S.

(**D9**) **Definition.** An n - *simplex* is the convex hull of a set S with

$$\# S = n + 1, \quad dim\ S = n.$$

We prove first that a simplex has a nonempty relative core.

(**L8**) **Lemma.** Let $\Delta_n = \text{conv}\{x_1, \ldots, x_{n+1}\}$ be an n-simplex. Then

$$\textbf{relcore}\ \Delta_n \neq \emptyset \tag{6.3}$$

Proof. By induction on n. For $n = 1$ the result follows from the order axiom (A7).

Assume the lemma holds for (n-1)-simplices. We denote

$$\Delta_{n-1} = \text{conv}\ \{x_1, \ldots, x_n\},$$

which is an (n-1)-simplex by (A4). We prove (6.3) by showing that if

$$x_0 \in \textbf{relcore}\ \Delta_{n-1}$$

then

$$(x_0, x_{n+1}) \subset \textbf{relcore}\ \Delta_n.$$

Indeed, let $x \in (x_0, x_{n+1})$, $y \in \mathbf{a}(\Delta_n)$ and let P be the plane $\mathbf{a}\{x_{n+1}, x, y\}$. Since $dim\ \mathbf{a}(\Delta_{n-1}) = dim\ \mathbf{a}(\Delta_n) - 1$, it follows from (A5) that P intersects $\mathbf{a}(\Delta_{n-1})$ in a line, say L.

Since $x_0 \in \textbf{relcore}\ \Delta_{n-1}$, there are points u, v in L such that $x_0 \in (u, v)$. We assume, without loss of generality, that the points y, v are on the same side of the line $\overline{x_0 x_{n+1}}$ in P. Then, by (A9) applied to $\Delta x_0 v x_{n+1}$, it follows that $\overline{xy}$ intersects either (v, x_{n+1}) or (x_0, v) at a point z. In either case $z \in \Delta_n$ and therefore $(x, z) \subset \Delta_n$, showing that

$$x \in \textbf{relcore}\ \Delta_n$$

$\square$

(**L9**) **Lemma.** If C is a nonempty convex set, $dim\ C < \infty$, then

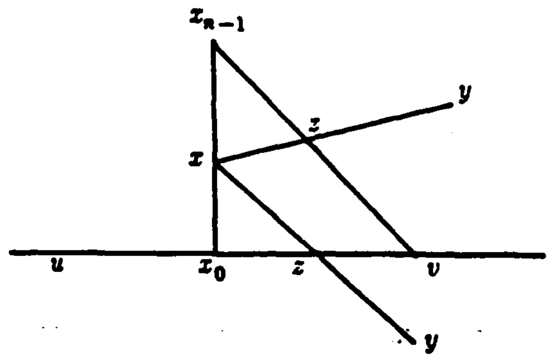

Fig. 6.6 Illustration of (L8)

$$dim\ C = dim\ \Delta_{max}$$

where Δ_{max} is a maximal dimensional simplex contained in C.

Proof. A maximal Δ_{max} can be constructed since C is finite-dimensional. We prove $\mathbf{a}(C) = \mathbf{a}(\Delta_{max})$.

From $\Delta_{max} \subset C$ it follows that

$$\mathbf{a}(\Delta_{max}) \subset \mathbf{a}(C).$$

If $\mathbf{a}(\Delta_{max}) \neq \mathbf{a}(C)$ then for any $x \in C \ / \ \mathbf{a}(\Delta_{max})$,

$$\mathbf{conv}\ \{\Delta_{max} \bigcup x\}$$

is a simplex of dimension $dim\ \Delta_{max} + 1$, by (A4), and contained in C. This contradicts the maximality of Δ_{max}.

$\square$

(T5) **Theorem.** If C is a nonempty convex set, $dim\ C < \infty$, then

$$\mathbf{relcore}\ C \neq \emptyset.$$

Proof. Follows from (L8) and (L9).

$\square$

7. SEPARATION

The main result here is Theorem (T7), stating conditions under which

any two distinct convex sets can be separated by a hyperplane.

From (T1) it follows that any hyperplane separates the space in the sense of (D2). We elaborate on this statement in the following:

(L10) Lemma. Given a hyperplane H in $\mathbf{X}$, there exist unique nonempty convex sets H^+, H^- such that

(a) H, H^+, H^- are disjoint, and

(b) $\mathbf{X} = H \bigcup H^+ \bigcup H^-$.

Proof. If dim $\mathbf{X} = 1$ then H is a point, and the lemma follows from (A8). Assume dim $\mathbf{X} \geq 2$, and let x_0 be any (fixed) point not in H. Define H^+, H^- by:

$$H^+ = \{y \notin H: (x_0,\ y) \bigcap H = \emptyset\}. \tag{7.1}$$

and

$$H^- = \{y \notin H: (x_0,\ y) \bigcap H \neq \emptyset\}. \tag{7.2}$$

Then (a) and (b) are obvious. We now prove the remaining statements.

Nonemptiness. Let h be any point in H. Then $(x_0,\ h) \subset H^+$, for otherwise $(x_0,\ h) \bigcap H \neq \emptyset$, and consequently the line $\overline{x_0 h} \subset H$, a contradiction. Similarly, $(h,\ y_0) \subset H^-$ for any $y_0 \in \overline{x_0 h}$ such that $h \in (x_0,\ y_0)$, see Fig. 7.1.

Convexity. Let $y_1,\ y_2 \in H^+$ and $y \in (y_1,\ y_2)$.

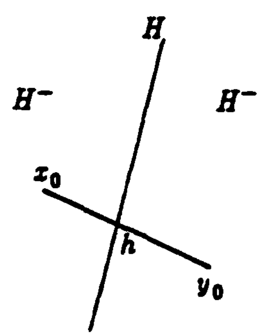

Fig. 7.1

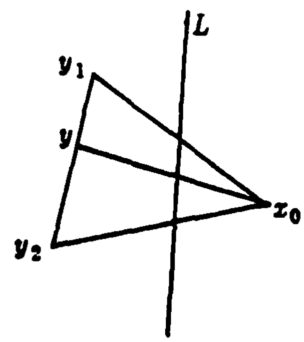

Fig. 7.2

If $y \in H^-$ then $(y, x_0) \cap H \neq \emptyset$ and we can use (A9) to show that

$$(y_1, x_0) \cap L \neq \emptyset \quad or \quad (y_2, x_0) \cap L \neq \emptyset.$$

where $L = H \cap a\{x_0, y_1, y_2\}$ is a line by (A5). see Fig. 7.2.

Thus $y_1 \in H^-$ or $y_2 \in H^-$, a contradiction.

$\therefore$ H^+ is convex.

The convexity of H^- is similarly proved.

Uniqueness. We show that the (unordered) pair $\{H^+, H^-\}$ is independent of the particular x_0 used in (7.1),(7.2).

Indeed, let $x_1 \in H^+$ and define

$$H_1^+ = \{y \notin H : (x_1, y) \cap H = \emptyset\}$$

and

$$H_1^- = \{y \notin H : (x_1, y) \cap H \neq \emptyset\}.$$

It suffices to show that $H^+ = H_1^+$.

Let $y \in H_1^+$, $y \notin H^+$. Then there is a point $h \in H \cap (x_0, y)$.

By (A9) and (A5) it follows then that

$$H \cap (x_0, x_1) \neq \emptyset$$

or

$$H \cap (x_1, y) \neq \emptyset$$

both contradictory. $\therefore$ $H_1^+ \subset H^+$. Reversing the roles of x_0 and x_1, we prove

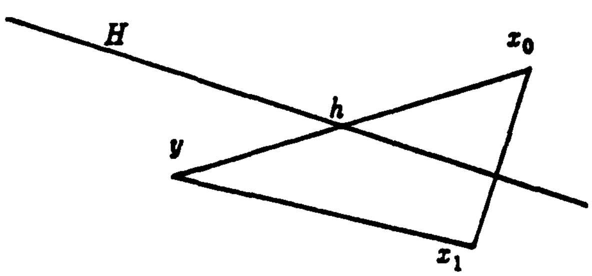

Fig. 7.3

the reverse inclusion, establishing $H^- = H_1^-$.

□

One can similarly obtain:

(L11) Lemma. Let H, H^+, H^- be as in (L10). Then

(i) $H^+ = \mathbf{core}\ H^+$, $H^- = \mathbf{core}\ H^-$

(ii) $H = \mathbf{lina}\ H^+ \bigcap \mathbf{lina}\ H^-$

(iii) $H \bigcup H^+ = \mathbf{cl}\ H^+$, $H \bigcup H^- = \mathbf{cl}\ H^-$.

□

(D10) Definitions. A convex set is

(i) *open* if $C \neq \mathbf{core}\ C$,

(ii) *relatively open* if $C = \mathbf{relcore}\ C$,

(iii) *closed* if $C = \mathbf{cl}\ C$.

The following definitions are justified by (L11) and (D10).

(D11) Definitions. Let H, H^+, H^- be as in (L10). Then

(i) H^+, H^- are called the *open halfspaces* corresponding to H.

(ii) $H \bigcup H^+$, $H \bigcup H^-$ are called the *closed halfspaces* corresponding to H.

(D12) Definitions. Let A, B be subsets of $\mathbf{X}$, and let H be a hyperplane. Then

(i) H *separates* A and B if A and B are contained in opposite closed halfspaces corresponding to H.

(ii) Moreover, if

$$A \bigcup B \text{ is not a subset of } H$$

then H *separates* A and B *properly*.

The following lemma implies the following converse of (T1): The only affine sets with the separation property (definition (D2)) are hyperplanes.

(**L12**) **Lemma.** If A, B, C are affine sets. $A \subset B$ and $A \neq B$. and if B separates C, then A does not separate C.

Proof. Suppose A separates C and let A^{-}. A^{-} be the "opposite sides" of A in C, i.e.

$$C = A \bigcup A^{-} \bigcup A^{-} \tag{7.3}$$

where A, A^{-}, A^{-} are disjoint,

x, y in A^{+} or in A^{-} $\Longrightarrow$ $(x, y) \bigcap A = \emptyset$,

$x \in A^{+}$, $y \in A^{-}$ $\Longrightarrow$ $(x, y) \bigcap A \neq \emptyset$.

Given that B separates C, let

$$C = B \bigcup B^{+} \bigcup B^{-} \tag{7.4}$$

be the analogous decomposition of C with respect to B.

Now let $x \in B^{+}$. Then $x \notin A$, and without loss of generality let $x \in A^{+}$. Any other point in B^{+} must also be in A^{+} for if $x \neq y \in B^{+}$, $y \in A^{-}$ then (x, y) intersects A but not B, a contradiction. Therefore

$$B^{+} \subset A^{+} \tag{7.5}$$

and similarly,

$$B^{-} \subset A^{-}. \tag{7.6}$$

Now $B^{+} = A^{+}$ and $B^{-} = A^{-}$ is impossible, since then (7.3) and (7.4) imply that $A = B$, a contradiction

Let $x \in A^{+} / B^{+}$. Then $x \notin B^{-}$ by (7.5), and therefore

$$x \in B.$$

Let $u \in B^{+}$ and let v be a point in $B^{-} \bigcap \overline{xu}$, see Fig. 7.4.

Then $u \in A^{+}$, $v \in A^{-}$, by (7.5),(7.6) and therefore (u, v) meets A at a point, say z. Since $x \notin A$, $z \neq x$.

The line $\overline{zx}$ is contained in B, by (C1), and therefore,

$u \in B$, a contradiction.

□

(**D13**) **Definition.** A *convex pair* in $\mathbf{X}$ is an unordered pair $\{C, D\}$ of nonempty convex sets with

$$\mathbf{X} = C \bigcup D, \qquad C \bigcap D = \emptyset. \tag{7.7}$$

A classical result of Mazur [15] (see also [20] theorem 2.3 and references therein) can be stated for OIG as follows:

(**T6**) **Theorem.** If A, B are disjoint convex sets in $\mathbf{X}$, then there exists a convex pair $\{C, D\}$ with

$$A \subset C, \quad B \subset D. \tag{7.8}$$

Proof. We first prove the following auxilliary result:

(i) If $\emptyset \neq S$ is a convex set, and $x_0 \notin S$, then the set

$$K(S, x_0) = \{x : x \in [x_0, y], y \in S\} \tag{7.9}$$

is convex. Here we denote by $[a, b]$ the *closed segment* joining a, b, defined by

$$[a, b] = (a, b) \bigcup \{a\} \bigcup \{b\}. \tag{7.10}$$

Proof of (i). Let $x_1, x_2 \in K(S, x_0)$ i.e. there are $y_1, y_2 \in S$ such that

$$x_i \in [x_0, y_i], \quad i = 1, 2.$$

Let $x_3 \in (x_1, x_2)$. By Lemma (L4), applied to $\Delta x_0 y_1 y_2$ (see Fig. 7.5), the line

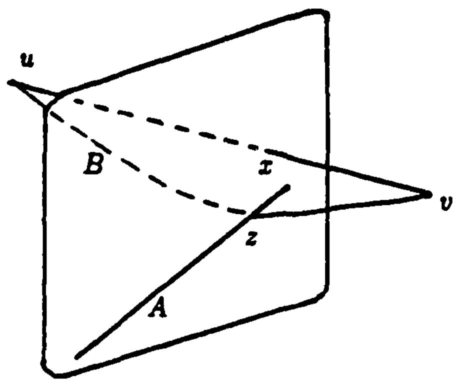

Fig 7.4 Proof of (L12)

$\overline{x_0 x_3}$ intersects (y_1, y_2) at a point, say y_3. Then $y_3 \in S$ and

$$x_3 \in [x_0, y_3] \subset K(S, x_0)$$

Having established (i), the classical proof ([20], Theorem 2.3) can be adapted, invoking Zorn's Lemma to obtain the pair $\{C, D\}$ as a maximal element of the set (partially ordered by inclusion) of disjoint convex sets $\{C, D\}$ satisfying (7.8).

□

A hyperplane separating disjoint convex sets A, B will be given, in terms of the convex pair $\{C, D\}$ of (T6), in Theorem (T7) below. First we require:

(D14) Definition. An affine set A is *openly decomposable* if A is the union of two disjoint, relatively open, nonempty convex sets.

(D15) Definition. A geometry $\underline{G} = \{\mathbf{X}, \underline{\mathbf{A}}, dim\}$ is a *Complete Ordered Incidence Geometry* (*COIG* for short) if it satisfies, in addition to Axioms (A1) - (A9), the following

(A10) Completeness Axiom. No line in $\underline{G}$ is openly decomposable.

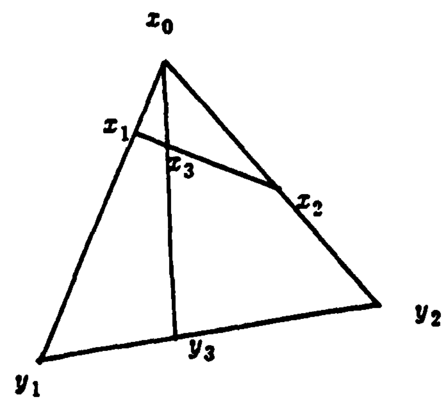

Fig. 7.5 Proof of (T6)

This version of *completeness* agrees with its usage in algebra. It can be shown to be equivalent to the completeness defined and used in [5].

An example of a non-complete geometry is:

(**M8**) **The rational Euclidean n-dimensional space.** Here $\mathbf{X} = \mathcal{Q}^n$, the set of rational n-tuples, and $\underline{\mathbf{A}}$ and *dim* agree with their standard vector space meanings.

We need the following property of a geometry $\underline{\mathbf{G}}$ (seemingly stronger than competeness): *No affine set* (of any dimension) *is openly decomposable.* The next lemma shows this to be equivalent to completeness.

(**L13**) **Lemma.** Let $\underline{\mathbf{G}} = \{\mathbf{X}, \underline{\mathbf{A}}, dim\}$ be a COIG. Then no affine set is openly decomposable.

Proof. Let $A \in \underline{\mathbf{A}}$ be openly decomposable, i.e.

$$A = C_1 \bigcup C_2 \tag{7.11}$$

where C_1, C_2 are disjoint, nonempty, relatively open, convex sets.

From (7.11) it follows that

$$A = \mathbf{a}(C_1) \bigcup \mathbf{a}(C_2)$$

and consequently that

$$A = \mathbf{a}(C_1) = \mathbf{a}(C_2)$$

showing that, restricted to A, the relative cores of C_1 and C_2 can be taken as cores, i.e.

$$C_1 = \textbf{core } C_1, \quad C_2 = \textbf{core } C_2. \tag{7.12}$$

Choose any two points $x_1 \in C_1$ and $x_2 \in C_2$, and let L be the line $\overline{x_1 x_2}$. From (7.12) follows the existence of two points z_1, z_2 such that

$$(x_i, z_i) \subset L \qquad i = 1, 2.$$

Extending the two (relatively) open segments (x_i, z_i) beyond x_i, ($i = 1, 2$), we get the (unbounded on one side) intervals:

$$I_i = (x_i, z_i) \bigcup \{y \in L : x_i \in (z_i, y)\} \qquad i = 1, 2$$

By Zorn's Lemma, the set of such intervals has a maximal element $\{I_1, I_2\}$, and consequently $L = I_1 \bigcup I_2$ showing that L is openly decomposable, violating (**A10**).

□

The following lemma gives conditions under which we can associate with any convex pair $\{C, D\}$ a hyperplane H having $\{\mathbf{core}\ C, \mathbf{core}\ D\}$ as its opposite sides.

(L14) Lemma. If $\{C, D\}$ is a convex pair in $\mathbf{X}$, then the set H defined by

$$H = \mathbf{cl}\ C \bigcap \mathbf{cl}\ D \tag{7.13}$$

satisfies:

(a) $H \bigcap \mathbf{core}\ C = \emptyset = H \bigcap \mathbf{core}\ D$

(b) If the geometry is complete then $H \neq \emptyset$ and

$$X = H \bigcup \mathbf{core}\ C \bigcup \mathbf{core}\ D \tag{7.14}$$

(c) Moreover, if either $\mathbf{core}\ C \neq \emptyset$ or $dim\ \mathbf{X} < \infty$, then H is a hyperplane.

Proof.

(a) $H \bigcap \mathbf{core}\ C = \mathbf{cl}\ C \bigcap \mathbf{cl}\ D \bigcap \mathbf{core}\ C$

$= \mathbf{cl}\ D \bigcap \mathbf{core}\ C$

If $x \in \mathbf{core}\ C \bigcap \mathbf{cl}\ D$ then:

(i) For every $y \neq x$ there is a $z \in (x, y)$ such that

$(x, z) \subset C$, by (D7)(i)

(ii) There is a $t \neq x$ such that $(t, x) \subset D$, by (D7)(iii,iv)

i.e. $(t, x) \bigcap C = \emptyset$, by (7.7) contradicting (a) for $y = t$.

Therefore $H \bigcap \mathbf{core}\ C = \emptyset$.

Similarly $H \bigcap \mathbf{core}\ D = \emptyset$.

(b) Since $\mathbf{X}$ is not openly decomposable, by (L13), we can choose an

$$x \notin \mathbf{core}\ C \bigcup \mathbf{core}\ D.$$

By (D7)(i) there is a y_1 such that

$$(x,\ y_1) \bigcap C = \emptyset$$

hence $(x,\ y_1) \subset D$ by (7.7).

Similarly there is a y_2 such that

$$(x,\ y_2) \bigcap D = \emptyset$$

i.e. $(x,\ y_2) \subset C$. But then, by (D7)(iii),

$$x \in \mathbf{lina} C \bigcap \mathbf{lina} D \subset \mathbf{cl} C \bigcap \mathbf{cl} D = H$$

(c) First we show that H is affine.

Let $x,\ y \in H$ and let z be on the line $\overline{xy}$. We show that $z \in H$, proving that $\mathbf{l}(H) = H$ and ,by (T2), that H is affine.

If $z \in (x,\ y)$ then $z \in H$ by (T4)(b).

Let $z \notin (x,\ y)$, and assume, by (A8), that $y \in (x,\ z)$.

If $z \notin H$, then we may take $z \in \mathbf{core}\ C$, by (b).

Therefore

$(x,\ z) \subset \mathbf{core}\ C$, by (T3)

and hence $y \in (x,\ z) \subset \mathbf{core}\ C$, by (T4)(a),

contradicting $y \in H$, by (a).

This completes the proof that H is an affine set. We show next that $H \neq \mathbf{X}$.

(i) If $\mathbf{core}\ C \neq \emptyset$ then $H \neq \mathbf{X}$ by (a).

(ii) If $\dim\ \mathbf{X} < \infty$ and if $H = \mathbf{X}$ then $\mathbf{cl}\ C = \mathbf{X}$.

Now $\dim\ C = \dim\ \mathbf{X}$ implies that $\mathbf{core}\ C \neq \emptyset$, by (T5), returning us to case (i).

Othewise, let $\dim\ C < \dim\ \mathbf{X}$. Then

$$\text{cl } C \subset \text{cl } \mathbf{a}(C) = \mathbf{a}\,(C) \neq \mathbf{X}.$$

a contradiction.

Finally. H separates $\mathbf{X}$. Indeed. by (a) and (b). **core** C and **core** D are the open halfspaces ((D11)(i)) corresponding to H. Combining (T1) and (L12) it follows that H is a hyperplane.

□

Combining the above results, we can finally state the

(T7) **Separation Theorem.** Let $\underline{\mathbf{G}} = \{\mathbf{X}\,,\, \underline{\mathbf{A}}\,,\, dim\}$ be a complete ordered-incidence geometry, and let A, B be disjoint convex sets in $\mathbf{X}$. Then there is a hyperplane H properly separating A and B if:

(a) **core** $A \neq \emptyset$, in which case

$$H \bigcap \textbf{core } A = \emptyset,$$

or if

(b) $dim\ \mathbf{X} < \infty$.

Proof. Let $\{C,\, D\}$ and H be given by (T6) and (L14) respectively. Then H separates A and B in the sense that

$$A \subset H \bigcup \textbf{core} C = H \bigcup H^{+} \tag{7.15}$$

$$B \subset H \bigcup \textbf{core} D = H \bigcup H^{-}$$

To prove proper eparation we must show, by (D12)(ii) that

$$A \bigcup B \text{ is not a contained in } H \tag{7.16}$$

(a) If **core**$A \neq \emptyset$ then, by (7.8),

$$\textbf{core } A \subset \textbf{core } C \neq \emptyset$$

and (7.16) follows from (L14)(a).

(b) Let $dim\ \mathbf{X} < \infty$, **core** $A = \emptyset$ (otherwise it is case (a) again), and

$$A \bigcup B \subset H$$

then we restrict the discussion to H which we denote by H_1. In H_1 there is a

hyperplane H_2 (i.e. $dim\ H_2 = dim\ H_1 - 1$) separating A and B in the sense of (7.15). Now there are two cases:

(i) H_2 separates A and B properly

(ii) $A \bigcup B \subset H_2$

In case (ii) we repeat the process: restrict to H_2, find a hyperplane H_3 (in H_2) separating A and B ,etc.

From

$$dim\ H_{i-1} = dim\ H_i - 1$$

it follows that after finitely many repetitions, an affine set H_i is reached in which one of the sets A, B has a nonempty core ,i.e.

$$dim\ H_i = \min\{dim\ A,\ dim\ B\}$$

and, by part (a), it is case (i),(although case (i) may occur sooner.)

Suppose then that case (i) is reached after k successive restrictions, a situation described by

$$\mathbf{X} = H_0 \supset H_1 \supset H_2 \dots \supset H_{k-1}$$

where H_{k+1} separates A and B properly in H_k. We reverse our steps now, constructing a sequence of affine sets

$$H_{k+1} = \overline{\overline{H}}_{k+1} \subset \overline{\overline{H}}_k \dots \subset \overline{\overline{H}}_2 \subset \overline{\overline{H}}_1 \subset \overline{\overline{H}}_0 = \mathbf{X} \tag{7.17}$$

where $\overline{\overline{H}}_{i+1}$ separates A and B properly in $\overline{\overline{H}}_i$, ($i = k, \cdots, 0$), and $\overline{\overline{H}}_1$ is then a hyperplane properly separating A and B in $\mathbf{X}$.

It is easily verified that a sequence (7.17) may be defined recursively as follows:

For i = k, ... , 1

choose any $x_i \in H_{i-1} / \overline{\overline{H}}_i$

define $\overline{\overline{H}}_{i-1} = \mathbf{a}\{\overline{\overline{H}}_i,\ x_i\}$

$\square$

Remarks.

(a) If core $A \neq \emptyset$, the assumption $A \bigcap B = \emptyset$ in (T7) can be replaced by

$$\text{core } A \cap B = \emptyset$$

(b) Note that the proof of (T7) uses the fact that no affine set is openly decomposable, since case (i) may arise at any dimension.

(c) The following example shows that completeness is needed in (T7).

(**E7**) **Example.** Consider the rational line Q^1 of (M8).

Then the sets

$$A = \{x \in Q^1 : x < \sqrt{2}\}$$
$$B = \{x \in Q^1 : x > \sqrt{2}\}$$

cannot be separated by a point (hyperplane) in Q^1.

8. APPLICATIONS TO FUNCTIONS ON THE REAL LINE: SUB-F FUNCTIONS AND FENCHEL DUALITY

In this section we use the terminology and notation of Section 3. Let F be a given *Beckenbach family* on the interval (a, b), see (D3), and let $\underline{G}_F$ be the associated *B - geometry,* described in (M6).

(**D16**) **Definitions.** (Beckenbach [1]) A function $f:(a, b) \to R$ is a *sub - F function* (*sub* - F for short) if for any two points

$$a < x_1 < x_2 < b$$

and $F_{12} \in$ F defined by

$$F_{12}(x_i) = f(x_i), \quad i = 1, 2$$

one has

$$f(x) \leq F_{12}(x) \qquad (x_1 \leq x < x_2) \tag{8.1}$$

f is *super*-F if we have the reverse inequality in (8.1).

Sub-F functions are generalizations of convex functions. Indeed, for the family F_1 of affine functions (see (E1)), *sub*-F_1 and *super*-F_1 become *convex* and *concave* (in the ordinary sense), respectively. The importance of sub-F

functions is evident, for example in their applications to 2nd order differential inequalities, see e.g. [6],[12] and [16].

In this section we study sub-F functions via their geometric properties (in $\underline{\mathbf{G}}_{\mathbf{F}}$.) We are able to establish useful analytical results, in particular a Fenchel duality theorem (T8) for sub-F functions.

As usual, we denote the *epigraph* and *hypograph* of f by

$$epi\ f = \{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \mu \geq f(x) \}, \tag{8.2}$$

$$hypo\ f = \{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \mu \leq f(x) \}. \tag{8.3}$$

As in the classical case, there is here a connection between the notions of *convexity of a function* (here *sub-F*) and *convexity of a set* (for the geometry $\underline{\mathbf{g}}_{\mathbf{F}}$, in the sense of (D5)(ii)). The proof of the following lemma is straightforward, hence omitted.

(L15) Lemma. A function $f:(a, b) \to R$ is:

(a) sub-F if and only if *epi* f is convex.

(b) super-F if and only if *hypo* f is convex.

The following characterization of the core of *epi* f is useful.

(L16) Lemma. Let a_1, b_1 be two points

$$a < a_1 < b_1 < b,$$

let f be sub-F on (a_1, b_1), and

$$A = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \begin{matrix} a_1 \leq x \leq b_1 \\ \mu \geq f(x) \end{matrix} \right\} \tag{8.4}$$

Then

$$\text{core } A = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \begin{matrix} a_1 < x < b_1 \\ \mu > f(x) \end{matrix} \right\} \tag{8.5}$$

Proof. We denote the right member of (8.5) by B.

core $A \subset B$. Otherwise it follows from the continuity of sub-F functions ([B], Theorem 6) that there is an

$$\begin{pmatrix} x_1 \\ \mu_1 \end{pmatrix} \in \text{core } A \tag{8.6}$$

such that

(i) $a_1 < x_1 < b_1, \;\; \mu_1 = f(x_1)$

or

(ii) $x_1 = a_1$

or

(iii) $x_1 = b_1$

In either case we contradict (8.6) by choosing an $\begin{pmatrix} x_2 \\ \mu_2 \end{pmatrix}$ such that the open segment [3]

$$\left(\begin{pmatrix} x_1 \\ \mu_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ \mu_2 \end{pmatrix} \right)$$

does not intersect A.

In case (i) we choose $\begin{pmatrix} x_2 \\ \mu_2 \end{pmatrix}$ with $\left\{ \begin{matrix} x_2 = x_1 \\ \mu_2 < \mu_1 \end{matrix} \right.$.

In cases (ii) and (iii), we choose $\begin{pmatrix} x_2 \\ \mu_2 \end{pmatrix}$ with $x_2 < a_1$ and $x_2 > b_1$,

[3] Recall from (M6) that the open segment

respectively.

<u>$B \subset \text{core } A$</u>. Let $\begin{pmatrix} x_1 \\ \mu_1 \end{pmatrix} \in B$ and let $\begin{pmatrix} x_2 \\ \mu_2 \end{pmatrix}$ be any point in $(a, b) \times R$.

We prove (8.6) by showing the existence of

$$\begin{pmatrix} x_3 \\ \mu_3 \end{pmatrix} \in \left(\begin{pmatrix} x_1 \\ \mu_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ \mu_2 \end{pmatrix} \right)$$

such that

$$\left(\begin{pmatrix} x_1 \\ \mu_1 \end{pmatrix}, \begin{pmatrix} x_3 \\ \mu_3 \end{pmatrix} \right) \subset A. \tag{8.7}$$

By (L15)(a), it suffices to consider

$$\begin{pmatrix} x_2 \\ \mu_2 \end{pmatrix} \notin A. \tag{8.8}$$

We begin with the case:

<u>$a_1 < x_2 < b_1$</u>

If $x_1 = x_2$, then $\mu_2 < f(x_1) < \mu_1$ by (8.5) and (8.8), and the (vertical) open segment $\left(\begin{pmatrix} x_1 \\ \mu_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ \mu_2 \end{pmatrix} \right)$ contains a point $\begin{pmatrix} x_3 \\ \mu_3 \end{pmatrix}$, $(x_3 = x_2 = x_1)$, satisfying (8.7).

If $x_1 \neq x_2$ consider the B-function F passing through

$$(x_i, f(x_i)) \quad i = 1, 2$$

and the B-function G passing through (see Fig. 8.1)

$$(x_i, \mu_i) \quad i = 1, 2.$$

$$\left(\begin{pmatrix} x_1 \\ \mu_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ \mu_2 \end{pmatrix} \right) = \begin{cases} \left\{ \begin{pmatrix} x_1 \\ \mu \end{pmatrix} : \mu \in (\mu_1, \mu_2) \right\} & \text{if } x_1 = x_2 \\ \left\{ \begin{pmatrix} x \\ F_{12}(x) \end{pmatrix} : x_1 < x < x_2 \right\} & \text{if } x_1 \neq x_2. \end{cases}$$

where F_{12} is determined by $F_{12}(x_i) = \mu_i, \quad i = 1,2$

Since $F(x_2) > G(x_2)$ and $F(x_1) < G(x_1)$ it follows that F and G intersect at a point $\begin{pmatrix} x_3 \\ \mu_3 \end{pmatrix}$ in A. which satisfies (8.7).

In the remaining cases ($x_2 < a_1$ or $x_2 > b_1$), similar arguments can be used.

□

Remark. The separation theorem (T7), applied to the sets (in R^2)

$$A = \text{core}\ (epi\ f) \text{ and } B = \left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix} \right\}$$

where x is a fixed point in (a, b), shows that a sub-F function has a *support* at each point in (a, b). This result is stated in [16] and applied to 2nd order differential inequalities. Indeed, the support property in necessary and sufficient for f to be sub-F.

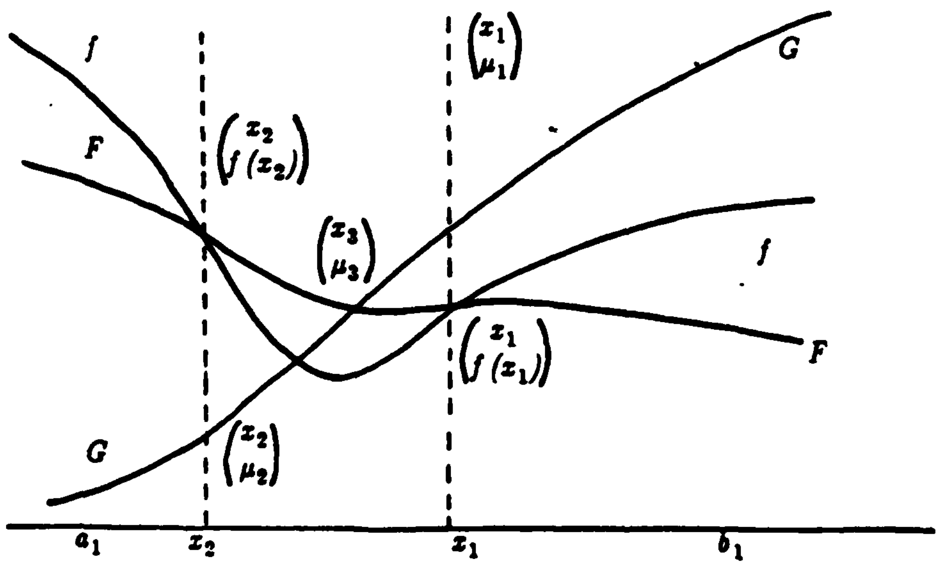

Fig. 8.1

From here on we specialize to the B-families of (E5).

$$\mathbf{F} = \{F(x) = \phi(\alpha, x) - \beta : \alpha, \beta \in R\} \tag{8.9}$$

where ϕ is differentiable in α, x and

$$\frac{\partial \phi}{\partial \alpha} \text{ an increasing function of } x. \tag{8.10}$$

As in [2] we define the *dual family*

$$\mathbf{F}^* = \{F^*(\alpha) = \phi(\alpha, x) - \beta : x, \beta \in R\} \tag{8.11}$$

which is a B-family if

$$\frac{\partial \phi}{\partial x} \text{ is an increasing function of } \alpha. \tag{8.12}$$

Note that in $\mathbf{F}^*$ the *argument* is α (one of the parameters of $\mathbf{F}$) and the *parameters* are x, β. Thus any pair $\{x, \beta\}$ determines a unique $F^* = F^*(\cdot \, ; x, \beta)$ in $\mathbf{F}^*$. The conditions (8.10) and (8.12) guarantee that both $\mathbf{F}$ and $\mathbf{F}^*$ are B-families.

As in convex analysis [19], we denote the *effective domain* of a function f by *dom* f.

(D17) Definitions. Given a function $f : (a, b) \to R$,

(i) the *(convex) conjugate* of f, f^*, is

$$f^*(\alpha) = \sup_{x \in dom\, f} \{\phi(\alpha, x) - f(x)\} \tag{8.13}$$

(ii) the *(concave) conjugate* of f, f_*, is

$$f_*(\alpha) = \inf_{x \in dom\, f} \{\phi(\alpha, x) - f(x)\} \tag{8.14}$$

In convex analysis, the conjugate f^* is always convex, regardless of f. The analogous result here is:

(L17) Lemma. For any function $f : (a, b) \to R$,

(a) the (convex) conjugate f^* is sub-$\mathbf{F}^*$,

(b) the (concave) conjugate f_* is super-$\mathbf{F}^*$.

Proof. (a) Follows from (L15)(a) once we show that the set $epi\ f^*$ is convex in the geometry $\underline{\mathbf{G}}_{F^*}$, determined by the dual family $\mathbf{F}^*$. Now

$$epi\ f^* = \left\{ \begin{pmatrix} \alpha \\ \mu \end{pmatrix} : \mu \geq \sup_x \{\phi(\alpha, x) - f(x)\} \right\},$$

$$= \left\{ \begin{pmatrix} \alpha \\ \mu \end{pmatrix} : \mu \geq \sup_{\substack{x,\beta \\ \beta \geq f(x)}} \{\phi(\alpha, x) - \beta\} \right\},$$

$$= \left\{ \begin{pmatrix} \alpha \\ \mu \end{pmatrix} : \mu \geq \phi(\alpha, x) - \beta, \textit{ for all } x, \beta \textit{ such that } \beta \geq f(x) \right\}$$

$$= \bigcap_{\substack{x,\beta \\ \beta \geq f(x)}} \left\{ \begin{pmatrix} \alpha \\ \mu \end{pmatrix} : \mu \geq \phi(\alpha, x) - \beta \right\}$$

so that, by (8.12),

$$epi\ f^* = \bigcap \{epi\ F^* : F^* \in \text{a subset of } \mathbf{F}^*\}$$

Since each F^* is sub-$\mathbf{F}^*$, it follows from (L15)(a) that

$$epi\ F^* \text{ is convex (in } \underline{\mathbf{G}}_{\mathbf{F}^*}), \forall F^* \in \mathbf{F}^*$$

and therefore $epi\ f^*$ is convex in $\underline{\mathbf{G}}_{\mathbf{F}^*}$.

(b) Follows similarly from (L15)(b).

□

A duality theorem of Fenchel type (see also [19], Theorem 31.1) now follows. A (somewhat weaker) Fenchel duality theorem was proved in [2] for F-convex functions:$R^n \to R$.

(T8) Theorem. Let

f be a sub-F function :$(a, b) \to R$

g be a super-F function :$(a, b) \to R$

and consider the pair of problems[4]

$$(\mathbf{P}) \quad \inf \{ f(x) - g(x) : x \in dom\ f \cap dom\ g \}$$

$$(\mathbf{D}) \quad \sup \{ g_*(\alpha) - f^*(\alpha) : \alpha \in dom\ f^* \cap dom\ g_* \}$$

If[5]

$$\operatorname{int} dom\ f \cap \operatorname{int} dom\ g \neq \emptyset \tag{8.15}$$

then

$$\inf(\mathbf{P}) = \max(\mathbf{D})$$

Proof. This proof is similar to Rockafellar's proof ([19], Theorem 31.1) for the classical case.

From (8.13) and (8.14),

$$g(x) + g_*(\alpha) \leq \phi(\alpha, x) \leq f(x) + f^*(\alpha), \quad \forall x, \alpha$$

so that

$$f(x) - g(x) \geq g_*(\alpha) - f^*(\alpha), \quad \forall \alpha, x$$

proving that

$$\inf(\mathbf{P}) \geq \sup(\mathbf{D}) \tag{8.16}$$

In particular,

$$\inf(\mathbf{P}) = -\infty \Rightarrow \sup(\mathbf{D}) = -\infty$$

Let $\inf(\mathbf{P}) > -\infty$, and denote

$$\begin{aligned} \gamma &= \inf(\mathbf{P}) \\ &= \sup\{\beta : f(x) \geq g(x) + \beta, \forall x\} \end{aligned} \tag{8.17}$$

By (8.16) it suffices to show that there exists an $\alpha \in dom\ f^* \cap dom\ g_*$ such that

$$g_*(\alpha) - f^*(\alpha) \geq \gamma \tag{8.18}$$

Define two sets

$$A = epi\ f$$

---

[4] The difference $f - g$ was shown in ([2], Theorem 4) to be a unimodal function.

[5] $dom\ f$ and $dom\ g$ are intervals in (a, b), and int denotes the interior of a real interval.

$$B = hypo\ (g+\gamma) = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \mu \le g(x) + \gamma \right\}$$

Note that for families **F** of type (8.9),

$$f \text{ is } \left\{ \begin{matrix} sub-\mathbf{F} \\ \\ super-\mathbf{F} \end{matrix} \right\} \Rightarrow (f+\gamma) \text{ is } \left\{ \begin{matrix} sub-\mathbf{F} \\ \\ super-\mathbf{F} \end{matrix} \right\}, \quad \forall \gamma \in R.$$

Therefore, using (L15), A and B are convex in $\underline{\mathbf{G}}_{\mathbf{F}}$.

From (L16) it follows that

$$\{\text{core } A\} \bigcap B = \emptyset$$

and by (T7) there is a hyperplane H separating core A and B, and therefore separating A and B.

In the geometry $\underline{\mathbf{G}}_{\mathbf{F}}$ hyperplanes are lines as defined in (M6). By (8.15), the separating line H cannot be vertical, and therefore is of the form:

$$H = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \mu = \phi(x, \alpha^*) - \beta^* \right\}$$

for some pair of parameters α^*, β^*.

Since H separates A and B,

$$f(x) \ge \phi(\alpha^*, x) - \beta^* \ge g(x) + \gamma, \quad \forall x.$$

$$\therefore \qquad \beta^* \ge \sup_x \{\phi(\alpha^*, x) - f(x)\} = f^*(\alpha^*)$$

$$\therefore \qquad \gamma + \beta^* \le \inf_x \phi(\alpha^*, x) - g(x)\} = g_*(\alpha^*)$$

And finally,

$$\gamma \le g_*(\alpha^*) - f^*(\alpha^*), \quad \text{proving (8.18).}$$

$\square$

The following example illustrates the validity of (T8). Here (P) is a convex program, and there are infinitely many possible Fenchel duals, corresponding to the various decompositions of the objective function in the form

$f - g$. and the choice of the underlying family **F**. One such dual is (**D**) below.

(**E8**) **Example.** Let the primal problem be

$$(\mathbf{P}) \qquad \inf_{x \geq 0} \; (e^{x} + e^{-x})$$

here the optimal solution is $x^* = 0$.

We choose

$$f(x) = e^x + e^{-x}$$

$$g(x) = -\delta(x|R^+) = \begin{cases} 0 & , \; x \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$dom\ f = R, \qquad dom\ g = R^+$$

Consider the family **F** of functions

$$F(x) = \cosh\,(\alpha + x) - \beta$$

Here $\mathbf{F} = \mathbf{F}^*$ and (8.9)-(8.12) are satisfied. Since $f \in \mathbf{F}$, f is sub-**F**. Also (since **F** consists of *convex* functions), the indicator function g is (strictly) super-**F**. The conjugates can be computed to give:

$$f^*(\alpha) = \begin{cases} -\sqrt{(e^{\alpha} - 2)(e^{-\alpha} - 2)} \text{ if } |\alpha| \leq \log 2 \\ \quad \infty \quad \text{otherwise} \end{cases}$$

$$g_*(\alpha) = \cosh\,(\max\,\{0, \alpha\})$$

so that

$$dom\ f^* = [-\log 2, \log 2], \quad dom\ g_* = R$$

and the dual program is

$$(\mathbf{D}) \qquad \sup_{|\alpha| \leq \log 2} \{\cosh\,(\max\,(0, \alpha)) + \sqrt{(e^{\alpha} - 2)(e^{-\alpha} - 2)}\,\}$$

It can be verified that the optimal solution of (**D**) is $\alpha^* = 0$, and as anticipated by (**T8**),

$$\inf(\mathbf{P}) = \sup(\mathbf{D}) = 2$$

9. THE THEOREMS OF RADON AND HELLY

The (closely related) classical theorems of Radon, Helly and Caratheodory (see e.g. [7]) hold also for OIG. We illustrate this by proving Radon's theorem, (**T9**) below, from which Helly's theorem, (T10), follows by a standard argument.

(**L18**) **Lemma.** If $S \subset \mathbf{X}$, $\# S = n + 2$, $dim\ S = n$, then there is a subset $T \subset S$, $\# T = n$, such that

$$S = T \bigcup x \bigcup y$$

$$\mathbf{a}(T) \bigcap [x, y] \neq \emptyset.$$

Proof. Such a T would necessarily have

$$dim\ T = n - 1 \text{ or } n - 2$$

by (C4) and (A4). Also x, y cannot both lie on $\mathbf{a}(T)$.

We prove the lemma by induction on n.

For $n = 1$, the lemma is just the order axiom (A8).

We prove it for $n = 2$, where its statement reads:

If $\# S = 4$, $dim\ \mathbf{a}(S) = 2$, then $\exists$ a line passing through 2 points of S and separating the remaining two points.

Let the points be x_i, i =1, 2, 3, 4.

If

$$\overline{x_1 x_2} \bigcap (x_3, x_4) \neq \emptyset$$

then the claimed T is $\{x_1, x_2\}$. Otherwise consider $\Delta x_1 x_2 x_3$ and $L = \overline{x_4 y}$ for some $y \in (x_1, x_2)$, see Fig 9.1.

By (A9), L intersects (x_2, x_3) *or* (x_1, x_3). Without loss of generality, let L intersect (x_2, x_3) at z. Consider now $\Delta x_1 y x_4$. The line $\overline{x_2 x_3}$ intersects the side (x_1, x_4), and therefore it also intersects the side (x_1, x_4).

$\therefore$ $\overline{x_2 x_3}$ separates x_1, x_4.

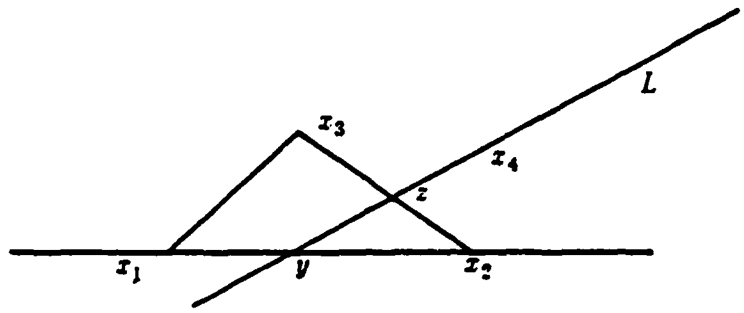

Fig. 9.1

Having established the lemma for $n = 2$, we assume its validity for all dimensions $\leq n - 1$, and prove it for dimension $= n$ (> 2).

Let

$$S = \{x_1, \cdots, x_{n+2}\}$$

and

$$\bar{S} = \{x_1, \cdots, x_{n+1}\}$$

There are now two possible cases:

(i) $dim\ \bar{S} = n$.

Since $\#\bar{S} = n + 1$, deleting any point from $\bar{S}$ would lower the dimension by 1. By the induction hypothesis, applied to the set

$$\{x_1, \cdots, x_n\}$$

there is a subset, say

$$T_1 = \{x_1, \cdots, x_{n-2}\}$$

such that

$$a(T_1) \cap [x_{n-1}, x_n] \neq \emptyset$$

The desired T is then

$$T = T_1 \bigcup x_{n+2}$$

(ii) $dim\ \bar{S} = n - 1$.

Again by the induction hypothesis, applied to $\bar{S}$, there is a subset, say

$$T_2 = \{x_1, \cdots, x_{n-1}\}$$

separating x_n, x_{n-1} and the desired T is

$$T = T_2 \bigcup x_{n+2}$$

□

(T9) The Radon Theorem. Let $S \subset \mathbf{X}$, $\# S \geq n + 2$, $dim\ S = n$. Then S can be partitioned into

$$S = S_1 \bigcup S_2$$

where

$$S_1 \bigcap S_2 = \emptyset$$

and

$$\text{conv } S_1 \bigcap \text{conv } S_2 \neq \emptyset.$$

Proof. Enough to consider the case $\# S = n + 2$.

We prove by induction on n.

For $n = 1$, the theorem follows from the order axioms.

The verification for $n = 2$ is as in the proof of (L18).

Assume the theorem holds for dimensions $\leq n - 1$, we will prove it for n (> 1).

Let

$$S = \{x_1, \cdots, x_{n+2}\}$$

By (L18) there is a subset, say

$$T = \{x_1, \cdots, x_n\}$$

such that

$$\mathbf{a}\,(T) \bigcap [x_{n+1}, x_{n+2}] \neq \emptyset$$

Let $y \in [x_{n+1}, x_{n+2}] \bigcap \mathbf{a}\,(T)$. Since $dim\ \mathbf{a}\,(T) \leq n - 1$, we use the induction hypothesis on $T \bigcup y$ to obtain two sets T_1, T_2 such that

$$T \bigcup y = T_1 \bigcup T_2, \qquad T_1 \bigcap T_2 = \emptyset$$

and

$$\text{conv } T_1 \bigcap \text{conv } T_2 \neq \emptyset$$

The sets S_1, S_2 can now be given as. say

$$S_1 = T_1 , \qquad S_2 = T_2 \bigcup x_{n-1} \bigcup x_{n-2}$$

□

The classical proof of [20] can now be used, verbatim, to obtain Helly's theorem from Radon's. We omit the details.

(T10) The Helly Theorem. Let S be a family of k convex sets, $k > n + 1$, in $\mathbf{X}$ where $dim\ \mathbf{X} = n$. If every $n + 1$ sets in S have a nonempty intersection, then S has a nonempty intersection.

□

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Geometry: axioms, incidence and order, convexity and inequalities, convex sets, convex analysis, generalized convexity, separation theorems, Radon and Helly theorems.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An Ordered Incidence Geometry, that is a geometry with certain axioms of incidence and order, is proposed as a minimal setting for the fundamental convexity theorems, such as the hyperplane separation theorem and the theorems of Radon and Helly. These theorems are usually stated, proved, understood and/or applied in the context of a linear vector space, but they require only incidence and order, (and for separation, completeness), and none of the linear structure of a vector space.

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